Topology and its Applications 54 (1993) 13–18
North-Holland

A construction of topological spaces rigid for continuous onto maps – an application of Shelah's Lemma

Jun Terasawa
Department of Mathematics, The National Defense Academy, Yokosuka 239, Japan

Received 30 October 1991
Revised 9 March 1992

Abstract

Terasawa, J., A construction of topological spaces rigid for continuous onto maps – an application of Shelah's Lemma, Topology and its Applications 54 (1993) 13–18.

Shelah's Lemma in the title refers to a generalization of a weaker form of the Lawrentieff Theorem.

A space X is called *rigid* for a class $\mathscr{F}$ of maps $X \to X$ if the identity 1_X is the only nonconstant map belonging to $\mathscr{F}$. It has been widely known that there are metric continua which are rigid for all maps into itself. Here, applying Shelah's Lemma, we show the following:

Theorem. *If a T_0-space X has weight $\leqslant \lambda$ and each of its nonempty open set has cardinality $\geqslant 2^\lambda$, then X contains a dense subspace A which is rigid for all maps onto itself.*

Keywords: T_0-space; Rigid; Autohomeomorphism; Weight; Dense subspace; Period; Orbit.

AMS (MOS) Subj. Class.: Primary 54G15; secondary 54C05.

Separation axioms are not assumed unless otherwise specified.

A space X is called *rigid* for a class $\mathscr{F}$ of continuous maps $X \to X$ into itself if the identity 1_X is the only nonconstant map belonging to $\mathscr{F}$. Following [4], let us call a space *strongly rigid* if it is rigid for the class of all maps into itself. Cook [1] showed that there is a metric continuum which is strongly rigid. Using different methods, Kannan and Rajagopalan [4] devised a construction scheme of connected strongly rigid nonregular T_2-spaces.

Correspondence to: Professor J. Terasawa, Department of Mathematics, The National Defence Academy, Yokosuka 239, Japan.

It should be noted that nonconnected spaces could never be strongly rigid. The purpose of the present paper is to show the existence of wider classes of (not necessarily connected) spaces which satisfy a somewhat weaker form of rigidity, that is, rigidity for the class of all maps *onto* itself. In fact, our argument achieves more, that is:

Theorem 1. *If a T_0-space X has weight $\leqslant \lambda$ and each of its nonempty open set has cardinality $\geqslant 2^\lambda$, then X contains a dense subspace A which is rigid for the class of all maps onto itself.*

Similarly, we can show

Theorem 2. *If a T_0-space X has weight $\leqslant \lambda$, has an autohomeomorphism h of finite period, and each of its nonempty open sets has cardinality $\geqslant 2^\lambda$, then X has a dense subspace A such that $h(A) = A$ and any onto map $f : A \to A$ is a homeomorphism of finite period satisfying $f(x) \in$ (orbit of x with respect to h) for every $x \in A$.*

Note that, generally speaking, the existence of an autohomeomorphism h implies the existence of vastly many maps besides h^i with $i \in \mathbb{Z}$. The situation could be illustrated in the following way. Suppose, for example, X is T_2, 0-dimensional and has no isolated points, and h has period 4. Then clearly there are three clopen sets U_i, $i = 1, 2, 3$, so that $h^i(U_j) \cap U_k = \emptyset$ for all $i, j, k = 0, 1, 2, 3$. Define $f : X \to X$ by

$$f(x) = \begin{cases} h(x), & \text{if } x \in U_1 \cup h^3(U_1) \cup \bigcup_{i=0}^{3} h^i(U_3), \\ h^2(x), & \text{if } x \in h(U_1) \cup U_2 \cup h^2(U_2), \\ x, & \text{otherwise}. \end{cases}$$

Then f is an autohomeomorphism of X, has points of period 1, 2, 3, 4, and is not of the form h^i.

1. Shelah's Lemma

Shelah [6, Lemma 2] essentially showed the following lemma. This had been shown only for complete metric spaces using Lawrentieff's Theorem [3, 4.3.20] and widely used in [2,7,8,9, etc.].

Lemma 3. *Suppose that topological spaces X, Y are given so that*

(1) *$w(Y) \leqslant \lambda$ and Y is T_0, and*

(2) *X has $\leqslant \mu$ many open sets.*

Then there is a collection $\{g_\alpha \mid \alpha < \mu^\lambda\}$ *of continuous maps from subsets of* X *to subsets of* Y *such that any map from a dense subspace of* X *onto a dense subspace of* Y *can be extended to some* g_α.

Proof. Let $\{V_\alpha \mid \alpha < \lambda\}$ be an open base of Y and $\mathscr{U}$ be the collection of all nonempty open sets of X. There are at most μ^λ many functions from λ to $\mathscr{U}$. Consider such a function $\{U_\alpha \mid \alpha < \lambda\}$ which satisfies that, for all α, β, $U_\alpha \cap U_\beta \neq \emptyset$ iff $V_\alpha \cap V_\beta \neq \emptyset$. For such a function, let G be the subset of the product set $X \times Y$ consisting of points (x, y) such that for all α, $x \in U_\alpha$ iff $y \in V_\alpha$.

There are at most μ^λ many such sets G. We claim that G is in fact a function from a subset X to a subset of Y, is continuous, and that every continuous map from a dense subspace of X onto a dense subspace of Y is extended to one of these G.

First, suppose that (x, y) and $(x, z) \in G$. If $y \neq z$, then there is an α such that $y \in V_\alpha \not\ni z$ or $y \notin V_\alpha \ni z$, because Y is a T_0-space. Suppose the former occurs. Then by the definition of G, we have $x \in U_\alpha$ and hence $z \in V_\alpha$. This is a contradiction. Thus (x, y) and $(x, z) \in G$ imply that $y = z$, and that G is a function. We can now write $G(x) = y$ in place of $(x, y) \in G$.

Now the definition of G easily implies that $G^{-1}(V_\alpha \cap (\text{Range } G)) = U_\alpha \cap (\text{Domain } G)$. This means that G is a continuous function.

Let $f: A \to B$ be any map from a dense subset A of X onto a dense subset B of Y. Since $V_\alpha \cap B \neq \emptyset$ is an open set of the subspace B, $f^{-1}(V_\alpha \cap B)$ is a nonempty open set of A, and hence, we can take an open set $U_\alpha \in \mathscr{U}$ such that $U_\alpha \cap A = f^{-1}(V_\alpha \cap B)$. It is easy to check that for all α, β, $U_\alpha \cap U_\beta \neq \emptyset$ iff $V_\alpha \cap V_\beta \neq \emptyset$, because A and B are dense subspaces of X and Y. Consider the above function of G for this $\{U_\alpha \mid \alpha < \lambda\}$. Then, for $x \in A$, we have $x \in U_\alpha$ iff $f(x) \in V_\alpha \cap B$ iff $f(x) \in V_\alpha$, and $G(x) = f(x)$. Thus G extends f. □

Quite similarly, the following can be shown. In fact, Shelah gave his lemma in this form for $\lambda = \omega$.

Lemma 4. *If* T_0*-spaces* X *and* Y *have weight* $\leqslant \lambda$, *then there is a collection* $\{g_\alpha \mid \alpha < 2^\lambda\}$ *of homeomorphisms from subspaces of* X *to subspaces of* Y *such that every homeomorphism from a dense subspace of* X *onto a dense subspace of* Y *can be extended to some* g_α.

2. Proof of Theorem 1

Let X be the space satisfying the condition of Theorem 1. Let $\{U_\xi \mid \xi < \lambda\}$ be an open base of X, and fix a collection $\mathscr{G} = \{g_\alpha \mid \alpha < 2^\lambda\}$ of continuous maps between subspaces of X so that every continuous map between dense subspaces of X can be extended to some g_α. The existence of this collection is assured by Lemma 3.

Let $\kappa: 2^\lambda \approx 2^\lambda \times \lambda \to \lambda$ be the natural projection. Then, for each $\xi < \lambda$, we have $|\kappa^{-1}(\xi)| = 2^\lambda$.

By transfinite induction on $\alpha < 2^\lambda$, we shall construct subsets A_α and B_α of X such that $\{A_\alpha\} \nearrow$, $\{B_\alpha\} \nearrow$, $A_\alpha \cap B_\alpha = \emptyset$ and $|A_\alpha \cup B_\alpha| \leqslant |\alpha| \cdot \omega$.

First of all, let $A_0 = B_0 = \emptyset$.

For a limit ordinal α, let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ and $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$.

Suppose that A_α and B_α have been defined. Then let $\mathscr{G}(\alpha) = \{g \in \mathscr{G} \mid \text{if } y \notin A_\alpha \text{ and } y \in \text{Range } g, \text{ then } g^{-1}(y) \subseteq B_\alpha \cup \{y\}\}$. If $g_\alpha \notin \mathscr{G}(\alpha)$, then there should exist a point $y \notin A_\alpha$ such that $y \in \text{Range } g_\alpha$ and $g_\alpha^{-1}(y) \not\subseteq B_\alpha \cup \{y\}$. Take any point $x \in g_\alpha^{-1}(y) \setminus B_\alpha \cup \{y\}$ and define $A'_\alpha = A_\alpha \cup \{x\}$ and $B'_\alpha = B_\alpha \cup \{y\}$. If $g_\alpha \in \mathscr{G}(\alpha)$, then let $A'_\alpha = A_\alpha$ and $B'_\alpha = B_\alpha$.

Moreover, let $\kappa(\alpha) = \xi$. Since $|U_\xi| \geqslant 2^\lambda$, we can take a point $z \in U_\xi \setminus A'_\alpha \cup B'_\alpha$. Now define $A_{\alpha+1} = A'_\alpha \cup \{z\}$ and $B_{\alpha+1} = B'_\alpha$. It is obvious that $A_{\alpha+1} \cap B_{\alpha+1} = \emptyset$ and $|A_{\alpha+1} \cup B_{\alpha+1}| \leqslant |\alpha| \cdot \omega$.

Finally define $A = \bigcup_{\alpha<2^\lambda} A_\alpha$ and $B = \bigcup_{\alpha<2^\lambda} B_\alpha$. Clearly we have that $A \cap B = \emptyset$.

For any $\xi < \lambda$ and $\alpha < 2^\lambda$, we can take $\beta > \alpha$ so that $\kappa(\beta) = \xi$. It follows that $U_\xi \cap A_{\beta+1} \setminus A_\beta \neq \emptyset$, and that $U_\xi \cap A \setminus A_\alpha \neq \emptyset$. That is, $A \setminus A_\alpha$ is a dense subset of X for any $\alpha < 2^\lambda$. Then A is also a dense subset of X.

Now let $f: A \to A$ be any onto continuous map. Then f can be extended to some g_α. If $g_\alpha \notin \mathscr{G}(\alpha)$, then there should exist a point $x \in A_{\alpha+1}$ such that $g_\alpha(x) \in B_{\alpha+1} \subseteq B$. This is a contradiction, because $x \in A$, $g_\alpha(x) = f(x) \in A$ and $A \cap B = \emptyset$ by construction.

So we have $g_\alpha \in \mathscr{G}(\alpha)$. Hence, if $y \notin A_\alpha$ and $y \in \text{Range } g_\alpha$, then $g_\alpha^{-1}(y) \subseteq B_\alpha \cup \{y\}$.

Let $y \in A \setminus A_\alpha$. Since $y \in \text{Range } f \subseteq \text{Range } g_\alpha$, we have $g_\alpha^{-1}(y) \subseteq B_\alpha \cup \{y\}$. Noting that g_α is the extension of f, $f^{-1}(y) \subseteq A$ and $A \cap B_\alpha = \emptyset$, we have $f^{-1}(y) = \{y\}$, that is, $f(y) = y$.

The density of $A \setminus A_\alpha$ implies that $f = 1_A$. Therefore our A is the required rigid space.

3. Proof of Theorem 2

Let p be the period of h and $\Omega(x) = \{h^i(x) \mid 0 \leqslant i < p\}$ denote the orbit of x with respect to h.

The first part of the proof goes quite similarly as above, except that $\mathscr{G}(\alpha)$ is now the set $\{g \in \mathscr{G} \mid \text{if } y \notin A_\alpha \text{ and } y \in \text{Range } g, \text{ then } g^{-1}(y) \subseteq B_\alpha \cup \Omega(y)\}$, and that $h(A_\alpha) = A_\alpha$, $h(B_\alpha) = B_\alpha$ and $A_\alpha \cap B_\alpha = \emptyset$ should be achieved at every stage. That is, if $g_\alpha \notin \mathscr{G}_\alpha$, then take $y \notin A_\alpha$ and $x \in g_\alpha^{-1}(y) \setminus B_\alpha \cup \Omega(y)$, and define $A'_\alpha = A_\alpha \cup \Omega(x)$, $B_{\alpha+1} = B'_\alpha = B_\alpha \cup \Omega(y)$, and, for $z \in U_\xi \setminus A'_\alpha \cup B'_\alpha$, $A_{\alpha+1} = A'_\alpha \cup \Omega(z)$.

So we have disjoint sets $A = \bigcup_{\alpha<2^\lambda} A_\alpha$ and $B = \bigcup_{\alpha<2^\lambda} B_\alpha$ in which $h(A) = A$, $h(B) = B$ and $A \setminus A_\alpha$ is dense in X.

Now let $f: A \to A$ be any onto continuous map. Then f is extended to some g_α. As above, we have $g_\alpha \in \mathscr{G}(\alpha)$ and hence, if $y \in A \setminus A_\alpha$ then $g_\alpha^{-1}(y) \subseteq B_\alpha \cup \Omega(y)$. It easily follows that $f^{-1}(y) \subseteq \Omega(y)$, because g_α extends f, $f^{-1}(y) \subseteq A$ and $A \cap B = \emptyset$. That is, we have $f^{-1}(\Omega(y)) \subseteq \Omega(y)$. Since $\Omega(y)$ is a finite set, this means each $f^{-1}(h^i(y))$ should be a singleton. In particular, each $f^{-1}(y)$ is a singleton. So, we have $f^{-1}(\Omega(y)) = \Omega(y)$. Therefore, for $x \in f^{-1}(A \setminus A_\alpha)$, we have $x = f^{-1}(f(x)) = h^i(f(x))$ for some i and hence that $h^{p-i}(x) = h^{p-i}(h^i(f(x))) = h^p(f(x)) = f(x)$. That is to say, for $x \in f^{-1}(A \setminus A_\alpha)$ we should have $f(x) \in \Omega(x)$.

We must show that $f^{-1}(A \setminus A_\alpha) = A \setminus A_\alpha$. To that end, first take $x \in f^{-1}(A \setminus A_\alpha)$. Then we have $x \notin A_\alpha$, because otherwise $f(x) \in \Omega(x) \subseteq \Omega(A_\alpha) = A_\alpha$. Thus $f^{-1}(A \setminus A_\alpha) \subseteq A \setminus A_\alpha$ is easily observed.

Conversely, suppose that $x \notin A_\alpha$ and $f(x) \in A_\alpha$. We will deduce a contradiction. Let $x = x_0$. Since $f(A) = A$, we can take a point $x_1 \in A$ in which $f(x_1) = x_0$. Since $x_1 \in f^{-1}(A \setminus A_\alpha)$, we have $x_1 \notin A_\alpha$ as seen above. $x_0 = f(x_1) \in \Omega(x_1)$ implies that $x_1 \in \Omega(x_0)$. Proceeding similarly, we take $x_{i+1} \in A$, for each $i \in \mathbb{N}$, so that $f(x_{i+1}) = x_i$. Since $x_{i+1} \in f^{-1}(A \setminus A_\alpha)$, we have $x_{i+1} \notin A_\alpha$ and $x_{i+1} \in \Omega(x_i) = \Omega(x_0)$. Since $\Omega(x_0)$ is a finite set, there should exist i, j such that $x_i = x_j$. Now take the smallest i so that $x_i = x_j$ for some $j > i$. If $i > 0$, then by construction we have $f(x_i) = x_{i-1}$ and $f(x_j) = x_{j-1}$, which is a contradiction. Hence $i = 0$ should hold. Then take the smallest $j > 0$ so that $x_0 = x_j$. This is a contradiction too, because $x_{j-1} = f(x_j) = f(x_0) = f(x) \in A_\alpha$.

Thus we have $f^{-1}(A \setminus A_\alpha) = A \setminus A_\alpha$. Therefore we can say that $f \mid A \setminus A_\alpha$ is injective and $f(x) \in \Omega(x)$ for any $x \in A \setminus A_\alpha$. Now, because $A \setminus A_\alpha$ is decomposed into finitely many sets, on each of which f coincides with some h^i, it is easy to observe that $f(x) \in \Omega(x)$ for all $x \in A$.

For $x \in A \setminus A_\alpha$, $f^2(x) = f(h^i(x)) = h^j(h^i(x)) = h^{i+j}(x)$ etc. holds. So, we have $\{f^n(x) \mid n > 0\} \subseteq \Omega(x)$. Since $\Omega(x)$ is a finite set, there are $n > m$ such that $f^n(x) = f^m(x)$. Injectivity of f on $A \setminus A_\alpha$ implies that $f^{n-m}(x) = x$, that is, $x \in \{f^n(x) \mid n > 0\}$. For each $x \in A \setminus A_\alpha$, take the smallest positive integer $n = n(x)$ such that $f^n(x) = x$. Then we have $\{f^n(x) \mid n \geqslant 0\} = \{f^i(x) \mid 0 \leqslant i \leqslant n(x)\} \subseteq \Omega(x)$, and hence $n(x) \leqslant p$.

Let $s = p!$. Then setting $s = n(x) \cdot r$, we have $f^s(x) = (f^{n(x)})^r(x) = x$ for all $x \in A \setminus A_\alpha$. Since $A \setminus A_\alpha$ is dense in A, this implies that $f^s(x) = x$ for all $x \in A$. Thus f is an autohomeomorphism on A of finite period.

References

[1] H. Cook, Continua which admit only the identity mapping onto non-degenerate subcontinua, Fund. Math. 60 (1967) 241–249.

[2] J. de Groot, Groups represented by homeomorphism groups, Math. Ann. 138 (1959) 80–102.

[3] R. Engelking, General Topology (PWN, Warsaw, 1977).

[4] V. Kannan and M. Rajagopalan, Construction and applications of rigid spaces I, Adv. in Math. 29 (1978) 89–130.

[5] T. Maćkowiak, The condensation of singularities in arc-like continua, Houston J. Math. 11 (1985) 535–558.
[6] S. Shelah, Decomposing topological spaces into two rigid homeomorphic subspaces, Israel J. Math. 63 (1988) 183–211.
[7] W. Sierpiński, Sur un problème concernant les types de dimensions, Fund. Math. 19 (1932) 65–71.
[8] J. Terasawa, A rigid space with many embeddings, Indag. Math. 49 (1987) 469–472.
[9] J. van Mill, A topological group having no homeomorphisms other than translations, Trans. Amer. Math. Soc. 280 (1983) 491–498.